A1 The largest such $k$ is $\lfloor \frac{n+1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$. For even $n$, this value is achieved by the partition

$$\{1, n\}, \{2, n-1\}, \ldots;$$

for odd $n$, it is achieved by the partition

$$\{n\}, \{1, n-1\}, \{2, n-2\}, \ldots.$$ 

One way to see that this is optimal is to note that the common sum can never be less than $n$, since $n$ itself belongs to one of the boxes. This implies that $k \leq (1 + \cdots + n)/n = (n+1)/2$. Another argument is that if $k > (n+1)/2$, then there would have to be two boxes with one number each (by the pigeonhole principle), but such boxes could not have the same sum.

Remark. A much subtler question would be to find the smallest $k$ (as a function of $n$) for which no such arrangement exists.

A2 The only such functions are those of the form $f(x) = cx + d$ for some real numbers $c, d$ (for which the property is obviously satisfied). To see this, suppose that $f$ has the desired property. Then for any $x \in \mathbb{R}$,

$$2f'(x) = f(x + 2) - f(x) = (f(x + 2) - f(x + 1)) + (f(x + 1) - f(x)) = f' + f'.$$

Consequently, $f'(x + 1) = f'(x)$.

Define the function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(x + 1) - f(x)$, and put $c = g(0)$, $d = f(0)$. For all $x \in \mathbb{R}$, $g'(x) = f'(x + 1) - f'(x) = 0$, so $g(x) = c$ identically, and $f'(x) = f(x + 1) - f(x) = g(x) = c$, so $f(x) = cx + d$ identically as desired.

A3 If $a = b = 0$, then the desired result holds trivially, so we assume that at least one of $a, b$ is nonzero. Pick any point $(a_0, b_0) \in \mathbb{R}^2$, and let $L$ be the line given by the parametric equation $L(t) = (a_0, b_0) + (a, b)t$ for $t \in \mathbb{R}$. By the chain rule and the given equation, we have $\frac{d}{dt}(h \circ L) = h \circ L$. If we write $f = h \circ L : \mathbb{R} \to \mathbb{R}$, then $f'(t) = f(t)$ for all $t$. It follows that $f(t) = Ce^t$ for some constant $C$. Since $|f(t)| \leq M$ for all $t$, we must have $C = 0$. It follows that $h(a_0, b_0) = 0$; since $(a_0, b_0)$ was an arbitrary point, $h$ is identically 0 over all of $\mathbb{R}^2$.

A4 Put

$$N = 10^{10^{10^n}} + 10^{10^n} + 10^n - 1.$$ 

Write $n = 2^mk$ with $m$ a nonnegative integer and $k$ a positive odd integer. For any nonnegative integer $j$,

$$10^{2^mj} \equiv (-1)^j \pmod{10^{2^m} + 1}.$$ 

Since $10^n \geq n \geq 2^m + m + 1$, $10^n$ is divisible by $2^m$ and hence by $2^{m+1}$, and similarly $10^{10^n}$ is divisible by $2^{10^m}$ and hence by $2^{10^{m+1}}$. It follows that

$$N = 1 + 1 + (-1) + (-1) \equiv 0 \pmod{10^{2^m} + 1}.$$ 

Since $N \geq 10^{10^n} > 10^n + 1 \geq 10^{2^m} + 1$, it follows that $N$ is composite.

A5 We start with three lemmas.

Lemma 1. If $x, y \in G$ are nonzero orthogonal vectors, then $x \times x$ is parallel to $y$.

Proof. Put $z = x \times y \neq 0$, so that $x, y$, and $z = x \times y$ are nonzero and mutually orthogonal. Then $w = x \times z \neq 0$, so $w = x \times z$ is nonzero and orthogonal to $x$ and $z$. However, if $(x \times x) \times y \neq 0$, then $w = x(x \times y) = (x \times x) \times y$ is also orthogonal to $y$, a contradiction. 

Lemma 2. If $x \in G$ is nonzero, and there exists $y \in G$ nonzero and orthogonal to $x$, then $x \times x = 0$.

Proof. Lemma 1 implies that $x \times x$ is parallel to both $y$ and $x \times y$, so it must be zero.

Lemma 3. If $x, y \in G$ commute, then $x \times y = 0$.

Proof. If $x \times y \neq 0$, then $x \times y = x \times y = -y \times x = -y \times x$, so $x \times y \neq y \times x$. 

We proceed now to the proof. Assume by way of contradiction that there exist $a, b \in G$ with $a \times b \neq 0$. Put $c = a \times b = a \times b$, so that $a, b, c$ are nonzero and linearly independent. Let $e$ be the identity element of $G$. Since $e$ commutes with $a, b, c$, by Lemma 3 we have $e \times a = e \times b = e \times c = 0$. Since $a, b, c$ span $\mathbb{R}^3$, $e \times x = 0$ for all $x \in \mathbb{R}^3$, so $e = 0$.

Since $b, c, b \times c = b \times c$ are nonzero and mutually orthogonal, Lemma 2 implies

$$b \times b = c \times c = (b \times c) \times (b \times c) = 0 = e.$$ 

Hence $b \times c = c \times b$, contradicting Lemma 3 because $b \times c \neq 0$. The desired result follows.
A6 First solution. Note that the hypotheses on $f$ imply that $f(x) > 0$ for all $x \in [0, +\infty)$, so the integrand is a continuous function of $f$ and the integral makes sense. Rewrite the integral as
\[
\int_0^\infty \left(1 - \frac{f(x+1)}{f(x)}\right) \, dx,
\]
and suppose by way of contradiction that it converges to a finite limit $L$. For $n \geq 0$, define the Lebesgue measurable set
\[
I_n = \{x \in [0, 1] : 1 - \frac{f(x+n+1)}{f(x+n)} \leq 1/2\}.
\]
Then $L \geq \sum_{n=0}^{\infty} \frac{1}{2}(1 - \mu(I_n))$, so the latter sum converges. In particular, there exists a nonnegative integer $N$ for which $\sum_{n=N}^{\infty} (1 - \mu(I_n)) < 1$; the intersection
\[
I = \bigcup_{n=N}^{\infty} I_n = [0, 1] - \bigcap_{n=N}^{\infty} ([0, 1] - I_n)
\]
then has positive Lebesgue measure.

By Taylor’s theorem with remainder, for $t \in [0, 1/2]$,
\[
-\log(1-t) \leq t + t^2 \sup_{t \in [0,1/2]} \left\{ \frac{1}{(1-t)^2} \right\} = t + \frac{4}{3}t^2 \leq \frac{5}{3}t.
\]
For each nonnegative integer $n \geq N$, we then have
\[
L \geq \sum_{n=N}^{\infty} \int_0^1 \left(1 - \frac{f(x+n+1)}{f(x+n)}\right) \, dx
\]
\[
\geq \sum_{n=N}^{\infty} \int_I \left(1 - \frac{f(x+n+1)}{f(x+n)}\right) \, dx
\]
\[
\geq \frac{3}{5} \sum_{n=N}^{\infty} \int_I \log \frac{f(x+n+1)}{f(x+n)} \, dx
\]
\[
= \frac{3}{5} \int_I \log \frac{f(x+N)}{f(x+n)} \, dx.
\]
For each $x \in I$, $\log f(x+N)/f(x+n)$ is a strictly increasing unbounded function of $n$. By the monotone convergence theorem, the integral $\int_I \log \frac{f(x+N)}{f(x+n)} \, dx$ grows without bound as $n \to +\infty$, a contradiction. Thus the original integral diverges, as desired.

Remark. This solution is motivated by the commonly-used fact that an infinite product $(1 + x_1)(1 + x_2)\cdots$ converges absolutely if and only if the sum $x_1 + x_2 + \cdots$ converges absolutely. The additional measure-theoretic argument at the beginning is needed because one cannot bound $-\log(1-t)$ by a fixed multiple of $t$ uniformly for all $t \in [0, 1]$.

Second solution. (Communicated by Paul Allen.) Let $b > a$ be nonnegative integers. Then
\[
\int_a^b \frac{f(x) - f(x+1)}{f(x)} \, dx = \sum_{k=a}^{b-1} \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+k)} \, dx
\]
\[
= \sum_{k=a}^{b-1} \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+k)} \, dx
\]
\[
\geq \sum_{k=a}^{b-1} \int_0^1 \frac{f(x+k) - f(x+k+1)}{f(x+a)} \, dx
\]
\[
= \int_0^1 \frac{f(x+a) - f(x+b)}{f(x+a)} \, dx.
\]
Now since $f(x) \to 0$, given $a$, we can choose an integer $l(a) > a$ for which $f(l(a)) < f(a+1)/2$; then $f(x+a) - f(x+l(a)) \geq 1 - f(l(a))/f(a+1) > 1/2$ for all $x \in [0, 1]$. Thus if we define a sequence of integers $a_n$ by $a_0 = 0$, $a_{n+1} = l(a_n)$, then
\[
\int_0^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx = \sum_{n=0}^{\infty} \int_0^{a_{n+1}} \frac{f(x) - f(x+1)}{f(x)} \, dx
\]
\[
> \sum_{n=0}^{\infty} \int_0^1 (1/2) \, dx,
\]
and the final sum clearly diverges.

Third solution. (By Joshua Rosenberg, communicated by Catalin Zara.) If the original integral converges, then on one hand the integrand $(f(x) - f(x+1))/f(x) = 1 - f(x+1)/f(x)$ cannot tend to 1 as $x \to \infty$. On the other hand, for any $a \geq 0$,
\[
0 < \frac{f(a+1)}{f(a)}
\]
\[
< \frac{1}{f(a)} \int_a^{a+1} f(x) \, dx
\]
\[
= \frac{1}{f(a)} \int_a^\infty (f(x) - f(x+1)) \, dx
\]
\[
\leq \int_a^\infty \frac{f(x) - f(x+1)}{f(x)} \, dx,
\]
and the last expression tends to 0 as $a \to \infty$. Hence by the squeeze theorem, $f(a+1)/f(a) \to 0$ as $a \to \infty$, a contradiction.

B1 First solution. No such sequence exists. If it did, then the Cauchy-Schwartz inequality would imply
\[
8 = (a_1^2 + a_2^2 + \cdots)(a_1^4 + a_2^4 + \cdots)
\]
\[
\geq (a_1^3 + a_2^3 + \cdots)^2 = 9,
\]
We thus cannot have $\varepsilon > 1$ because for any rational function $f$ for which $a_k^2 > 1$, we can find a positive integer $k$ for which $a_k^2 > 1$. However, in this case, for $m$ large, $a_k^{2m} > 2m$ and so $a_k^{2m} + 2m^2 + \cdots \neq 2m$.

**Third solution.** We generalize the second solution to show that for any positive integer $k$, it is impossible for a sequence of complex numbers to satisfy the given conditions in case the series $a_1^k + a_2^k + \cdots$ converges absolutely. This includes the original problem by taking $k = 2$, in which case the series $a_1^2 + a_2^2 + \cdots$ consists of nonnegative real numbers and so converges absolutely if it converges at all.

Since the sum $\sum_{i=1}^{\infty} |a_i|^k$ converges by hypothesis, we can find a positive integer $n$ such that $\sum_{i=n+1}^{\infty} |a_i|^k < 1$. For each positive integer $d$, we then have

$$kd - \sum_{i=1}^{n} a_i^{kd} \leq \sum_{i=n+1}^{\infty} |a_i|^{kd} < 1.$$ 

We thus cannot have $|a_1|, \ldots, |a_n| \leq 1$, or else the sum $\sum_{i=1}^{n} a_i^{kd}$ would be bounded in absolute value by $n$ independently of $d$. But if we put $r = \max\{|a_1|, \ldots, |a_n|\} > 1$, we obtain a contradiction because for any $\epsilon > 0$,

$$\lim_{d \to \infty} \sup_{d \to \infty} (r - \epsilon)^{-kd} \sum_{i=1}^{n} a_i^{kd} > 0.$$ 

For instance, this follows from applying the root test to the rational function

$$\sum_{i=1}^{n} \frac{1}{1 - a_i^{z}} = \sum_{d=0}^{\infty} \left( \sum_{i=1}^{n} a_i^{kd} \right) z^d,$$

which has a pole within the circle $|z| \leq r^{-1/k}$. (An elementary proof is also possible.)

**Fourth solution.** (Communicated by Noam Elkies.) Since $\sum_k a_k^2 = 2$, for each positive integer $k$ we have $a_k^2 \leq 2$ and so $a_k^2 \leq 2a_k^2$, with equality only for $a_k^2 \in \{0, 2\}$. Thus to have $\sum_k a_k^4 = 4$, there must be a single index $k$ for which $a_k^2 = 2$, and the other $a_k$ must all equal 0. But then $\sum_k a_k^{2m} = 2m \neq 2m$ for any positive integer $m > 2$.

**Remark.** Manjul Bhargava points out that more generally, a *Heronian triangle* (a triangle with integer sides and rational area) cannot have a side of length 1 or 2 (and again it is enough to treat the case of length 2). The original problem follows from this because a triangle whose vertices have integer coordinates has area equal to that of a triangle with integer vertices. This can only happen for $y = 0$, but then $A, B, C$ are collinear, a contradiction again.

**Remark.** Manjul Bhargava points out that more generally, a *Heronian triangle* (a triangle with integer sides and rational area) cannot have a side of length 1 or 2 (and again it is enough to treat the case of length 2). The original problem follows from this because a triangle whose vertices have integer coordinates has area equal to that of a triangle with integer vertices. This can only happen for $y = 0$, but then $A, B, C$ are collinear, a contradiction again.

**B2** The smallest distance is 3, achieved by $A = (0, 0), B = (3, 0), C = (0, 4)$. To check this, it suffices to check that $AB$ cannot equal 1 or 2. (It cannot equal 0 because if two of the points were to coincide, the three points would be collinear.)

The triangle inequality implies that $|AC - BC| \leq AB$, with equality if and only if $A, B, C$ are collinear. If $AB = 1$, we may assume without loss of generality that $A = (0, 0), B = (1, 0)$. To avoid collinearity, we must have $AC = BC$, but this forces $C = (1/2, y)$ for some $y \in \mathbb{R}$, a contradiction. (One can also treat this case by scaling by a factor of 2 to reduce to the case $AB = 2$, treated in the next paragraph.)

If $AB = 2$, then we may assume without loss of generality that $A = (0, 0), B = (2, 0)$. The triangle inequality implies $|AC - BC| \in \{0, 1\}$. Also, for $C = (x, y)$, $AC^2 = x^2 + y^2$ and $BC^2 = (2 - x)^2 + y^2$ have the same parity; it follows that $AC = BC$. Hence $c = (1, y)$ for some $y \in \mathbb{R}$, so $y^2$ and $y^2 + 1 = BC^2$ are consecutive perfect squares. This can only happen for $y = 0$, but then $A, B, C$ are collinear, a contradiction again.

**Remark.** Manjul Bhargava points out that more generally, a *Heronian triangle* (a triangle with integer sides and rational area) cannot have a side of length 1 or 2 (and again it is enough to treat the case of length 2). The original problem follows from this because a triangle whose vertices have integer coordinates has area equal to half an integer (by Pick’s formula or the explicit formula for the area as a determinant).

**B3** It is possible if and only if $n \geq 1005$. Since

$$1 + \cdots + 2009 = \frac{2009 \times 2010}{2} = 2010 \times 1004.5,$$

for $n \leq 1004$, we can start with an initial distribution in which each box $B_i$ starts with at most $i - 1$ balls (so
in particular \( B_1 \) is empty). From such a distribution, no moves are possible, so we cannot reach the desired final distribution.

Suppose now that \( n \geq 1005 \). By the pigeonhole principle, at any time, there exists at least one index \( i \) for which the box \( B_i \) contains at least \( i \) balls. We will describe any such index as being eligible. The following sequence of operations then has the desired effect.

(a) Find the largest eligible index \( i \). If \( i = 1 \), proceed to (b). Otherwise, move \( i \) balls from \( B_i \) to \( B_1 \), then repeat (a).

(b) At this point, only the index \( i = 1 \) can be eligible (so it must be). Find the largest index \( j \) for which \( B_j \) is nonempty. If \( j = 1 \), proceed to (c). Otherwise, move 1 ball from \( B_1 \) to \( B_j \); in case this makes \( j \) eligible, move \( j \) balls from \( B_j \) to \( B_1 \). Then repeat (b).

(c) At this point, all of the balls are in \( B_1 \). For \( i = 2, \ldots, 2010 \), move one ball from \( B_1 \) to \( B_i \) \( n \) times.

After these operations, we have the desired distribution.

B4 First solution. The pairs \((p, q)\) satisfying the given equation are those of the form \( p(x) = ax + b, q(x) = cx + d \) for \( a, b, c, d \in \mathbb{R} \) such that \( bc - ad = 1 \). We will see later that these indeed give solutions.

Suppose \( p \) and \( q \) satisfy the given equation; note that neither \( p \) nor \( q \) can be identically zero. By subtracting the equations

\[
\begin{align*}
p(x)q(x + 1) - p(x + 1)q(x) &= 1, \\
p(x - 1)q(x) - p(x)q(x - 1) &= 1,
\end{align*}
\]

we obtain the equation

\[
p(x)(q(x + 1) + q(x - 1)) = q(x)(p(x + 1) + p(x - 1)).
\]

The original equation implies that \( p(x) \) and \( q(x) \) have no common nonconstant factor, so \( p(x) \) divides \( p(x + 1) + p(x - 1) \). Since each of \( p(x + 1) \) and \( p(x - 1) \) has the same degree and leading coefficient as \( p \), we must have

\[
p(x + 1) + p(x - 1) = 2p(x).
\]

If we define the polynomials \( r(x) = p(x + 1) - p(x), s(x) = q(x + 1) - q(x) \), we have \( r(x + 1) = r(x) \), and similarly \( s(x + 1) = s(x) \). Put

\[
a = r(0), b = p(0), c = s(0), d = q(0).
\]

Then \( r(x) = a, s(x) = c \) for all \( x \in \mathbb{Z} \), and hence identically; consequently, \( p(x) = ax + b, q(x) = cx + d \) for all \( x \in \mathbb{Z} \), and hence identically. For \( p \) and \( q \) of this form,

\[
p(x)q(x + 1) - p(x + 1)q(x) = bc - ad,
\]

so we get a solution if and only if \( bc - ad = 1 \), as claimed.

Second solution. (Communicated by Catalin Zara.) Again, note that \( p \) and \( q \) must be nonzero. Write

\[
\begin{align*}
p(x) &= p_0 + p_1x + \cdots + p_mx^m, \\
q(x) &= q_0 + q_1x + \cdots + q_nx^n
\end{align*}
\]

with \( p_m,q_n \neq 0 \), so that \( m = \deg(p), n = \deg(q) \). It is enough to derive a contradiction assuming that \( \max\{m,n\} > 1 \), the remaining cases being treated as in the first solution.

Put \( R(x) = p(x)q(x+1)-p(x+1)q(x). \) Since \( m+n \geq 2 \) by assumption, the coefficient of \( x^{m+n-1} \) in \( R(x) \) must vanish. By easy algebra, this coefficient equals \( (m-n)p_mq_n \), so we must have \( m = n > 1 \).

For \( k = 1, \ldots, 2m-2 \), the coefficient of \( x^k \) in \( R(x) \) is

\[
\sum_{i+j > k, j > i} \binom{j}{k-i} \binom{i}{k-j} (p_iq_j - p_jq_i)
\]

and must vanish. For \( k = 2m-2 \), the only summand is \( (i,j) = (m-1,m) \), so \( p_{m-1}q_m = p_mq_{m-1} \).

Suppose now that \( h \geq 1 \) and that \( p_iq_j = p_jq_i \) is known to vanish whenever \( j > i \geq h \). (By the previous paragraph, we initially have this for \( h = m-1 \).) Take \( k = m+h-2 \) and note that the conditions \( i+j \geq h, j \leq m \) force \( i \geq h-1 \). Using the hypothesis, we see that the only possible nonzero contribution to the coefficient of \( x^k \) in \( R(x) \) is from \( (i,j) = (h-1,m) \). Hence \( p_{h-1}q_m = p_mq_{h-1} \); since \( p_m,q_m \neq 0 \), this implies \( p_{h-1}q_j = p_jq_{h-1} \) whenever \( j > h-1 \).

By descending induction, we deduce that \( p_iq_j = p_jq_i \) whenever \( j > i \geq 0 \). Consequently, \( p(x) \) and \( q(x) \) are scalar multiples of each other, forcing \( R(x) = 0 \), a contradiction.

Third solution. (Communicated by David Feldman.)

As in the second solution, we note that there are no solutions where \( m = \deg(p), n = \deg(q) \) are distinct and \( m + n \geq 2 \). Suppose \( p,q \) form a solution with \( m = n \geq 2 \). The desired identity asserts that the matrix

\[
\begin{pmatrix}
p(x) & p(x+1) \\
q(x) & q(x+1)
\end{pmatrix}
\]

has determinant 1. This condition is preserved by replacing \( q(x) \) with \( q(x) - tp(x) \) for any real number \( t \). In particular, we can choose \( t \) so that \( \deg(q(x) - tp(x)) < m \); we then obtain a contradiction.

B5 First solution. The answer is no. Suppose otherwise. For the condition to make sense, \( f \) must be differentiable. Since \( f \) is strictly increasing, we must have \( f'(x) \geq 0 \) for all \( x \). Also, the function \( f'(x) \) is strictly increasing: if \( y > x \) then \( f'(y) = f(f(y)) > f(f(x)) = f'(x) \). In particular, \( f'(y) > 0 \) for all \( y \in \mathbb{R} \).
For any \( x_0 \), if \( f(x_0) = b \) and \( f'(x_0) = a > 0 \), then 
\[ f'(x) > a \quad \text{for} \quad x > x_0 \] 
and thus \( f(x) \geq a(x - x_0) + b \) for \( x \geq x_0 \). Then either \( b < x_0 \) or \( a = f'(x_0) = f(f(x_0)) = f(b) \geq a(b - x_0) + b \). In the latter case, 
\[ b \leq a(x_0 + 1)/(a + 1) \leq x_0 + 1. \] 
We conclude in either case that \( f(x_0) \leq x_0 + 1 \) for all \( x_0 \geq -1 \).

It must then be the case that \( f(x) = f'(x) \leq 1 \) for all \( x \), since otherwise \( f(x) > x + 1 \) for large \( x \). Now by the above reasoning, if \( f(0) = b_0 \) and \( f'(0) = a_0 > 0 \), then \( f(x) > a_0 x + b_0 \) for \( x > 0 \). Thus for \( x > \max\{0, -b_0/a_0\} \), we have \( f(x) > 0 \) and \( f(f(x)) > a_0 x + b_0 \). But then \( f(f(x)) > 1 \) for sufficiently large \( x \), a contradiction.

**Second solution.** (Communicated by Catalin Zara.)

Suppose such a function exists. Since \( f \) is strictly increasing and differentiable, so is \( f \circ f = f' \). In particular, \( f \) is twice differentiable; also, \( f''(x) = f'(f(x))f'(x) \) is the product of two strictly increasing nonnegative functions, so it is also strictly increasing and nonnegative. In particular, we can choose \( \alpha > 0 \) and \( M \in \mathbb{R} \) such that \( f''(x) > 4\alpha \) for all \( x \geq M \). Then for all \( x \geq M \),
\[ f(x) \geq f(M) + f'(M)(x - M) + 2\alpha(x - M)^2. \]

In particular, for some \( M' > M \), we have \( f(x) \geq \alpha x^2 \) for all \( x \geq M' \).

Pick \( T > 0 \) so that \( \alpha T^2 > M' \). Then for \( x \geq T \), \( f(x) > M' \) and so \( f'(x) = f(f(x)) \geq \alpha f(x)^2 \). Now
\[ \frac{1}{f(T)} - \frac{1}{f(2T)} = \int_T^{2T} \frac{f'(t)}{f(t)^2} \, dt \geq \int_T^{2T} \alpha \, dt; \]
however, as \( T \to \infty \), the left side of this inequality tends to \( 0 \) while the right side tends to \( +\infty \), a contradiction.

**Third solution.** (Communicated by Noam Elkies.)

Since \( f \) is strictly increasing, for some \( y_0 \), we can define the inverse function \( g(y) \) of \( f \) for \( y \geq y_0 \). Then \( x = g(f(x)) \), and we may differentiate to find that
\[ 1 = g'(f(x))f'(x) = g'(f(f(x))f(f(x))). \] 
It follows that \( g'(y) = 1/f(y) \) for \( y \geq y_0 \); since \( g \) takes arbitrarily large values, the integral \( \int_{y_0}^{\infty} dy/f(y) \) must diverge. One then gets a contradiction from any reasonable lower bound on \( f(y) \) for \( y \) large, e.g., the bound \( f(x) \geq \alpha x^2 \) from the second solution. (One can also start with a linear lower bound \( f(x) \geq \beta x \), then use the integral expression for \( g \) to deduce that \( g(x) \leq \gamma \log x \), which in turn forces \( f(x) \) to grow exponentially.)

**B6** For any polynomial \( p(x) \), let \( [p(x)]A \) denote the \( n \times n \) matrix obtained by replacing each entry \( A_{ij} \) of \( A \) by \( p(A_{ij}) \); thus \( A^k = [x^k]A \). Let \( P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \) denote the characteristic polynomial of \( A \). By the Cayley-Hamilton theorem,
\[ 0 = A \cdot P(A) = A^{n+1} + a_{n-1}A^n + \cdots + a_0A \]
\[ = A^{m+1} + a_{m-1}A^m + \cdots + a_0A^{m+1-n} \]

Thus each entry of \( A \) is a root of the polynomial \( xp(x) \).

Now suppose \( m \geq n + 1 \). Then
\[ 0 = [x^{m+1-n}p(x)]A = A^{m+1} + a_{m-1}A^m + \cdots + a_0A^{m+1-n} \]

since each entry of \( A \) is a root of \( x^{m+1-n}p(x) \). On the other hand,
\[ 0 = A^{m+1-n} \cdot P(A) = A^{m+1} + a_{m-1}A^m + \cdots + a_0A^{m+1-n}. \]

Therefore if \( A^k = A^k \) for \( m + 1 - n \leq k \leq m \), then \( A^{m+1} = A^{m+1} \). The desired result follows by induction on \( m \).

**Remark.** David Feldman points out that the result is best possible in the following sense: there exist examples of \( n \times n \) matrices \( A \) for which \( A^k = A^k \) for \( k = 1, \ldots, n \) but \( A^{n+1} \neq A^{n+1} \).